Tutorial 8.

Preliminary:

Sinking-fund method: A loan calls n interest payments $L \cdot i$ and pay the single lump sum of amount L at time n.

Li payment in each period

$\frac{L}{s_{\overline{n}|j}}$ is level sinking fund deposit.

Outlay is $L(i+\frac{1}{s_{\overline{n}|j}})$.

Bond Price: P: price ; F: face amount ; C: redemption amount

r: coupon rate ; j: yield rate.

$$P = Cv_j^n + Fr(v_j + v_j^2 + \cdots + v_j^n) = Cv_j^n + Fr\,a_{\overline{n}|j}.$$

$$\begin{array}{cccc} 0 & 1 & 2 & \cdots & n \\ \hline & Fr & Fr & & Fr+C \end{array}$$

When $F=C$, $P = Fv_j^n + Fr\,a_{\overline{n}|j} \overset{v_j^n = 1-j\,a_{\overline{n}|j}}{=\!=\!=} F + F(r-j)a_{\overline{n}|j}$.

Exercise:

3.3.6s.

$L = 1000$, $n=10$, $i = 10\%$, $j = 14\%$

$L = P a_{\overline{n}|i} \Rightarrow P = 162.75$. ← another strategy.

Sinking fund deposits are $P - Li = 162.75 - 1000 \times 10\% = 62.75$.

Accumulate Value of sinking fund $AV = 62.75\, s_{\overline{n}|j} = 1213$.

Balance $1213 - 1000 = 213$.

three categories of financial instruments:

derivative: (options, futures)

equities: (stocks)

debt (bonds, mortgages).

4.1.4.

$r = \frac{6\%}{2} = 3\%, \quad j = \frac{5\%}{2} = 2.5\%, \quad n = 8 \times 2 = 16, \quad r' = \frac{5.5\%}{2} = 2.75\%.$

$C = F.$ $\quad P = F v_j^{n} + F r a_{\overline{n|}j} = F v_j^{2m} + F r' a_{\overline{2m|}j}$

$\Rightarrow \quad v_j^{n} + r a_{\overline{n|}j} = v_j^{2m} + r' a_{\overline{2m|}j}.$

$\Rightarrow \quad m = 21.5$ years.

4.1.12.

$F_1 = F_2 = 100, \quad P_1 + P_2 = 240, \quad r_1 = 2r_2, \quad P_1 - P_2 = 24. \qquad j = 3\%$

$\begin{cases} P_1 + P_2 = 240 \\ P_1 - P_2 = 24 \end{cases} \Rightarrow P_1 = 132, \; P_2 = 108.$

$P_1 = F_1 + F_1 (r_1 - j) a_{\overline{n|}j} \Rightarrow 132 = 100 + 100(2r_2 - j) a_{\overline{n|}j}$

$108 = 100 + 100(r_2 - j) a_{\overline{n|}j} \quad \Rightarrow \quad \frac{32}{8} = \frac{2r_2 - j}{r_2 - j}.$

$\Rightarrow r_2 = 0.0225, \; r_1 = 0.045.$

4.1.14.

$t = 25\%, \quad F = 1000, \quad r = 4\%, \quad n = 10, \quad j = 5\%.$

$P = C v_j^{10} + F r a_{\overline{n|}j}$

$\Rightarrow \quad P = 908.78.$

$C = 1000 - 0.25 \times \underbrace{(1000 - P)}_{\text{discount}}$

Tutorial 3.3.6S, 4.1.4, 4.1.12, 4.1.14 Problem set 3.3.1, 3.3.2, 3.3.3, 3.3.4, 3.4.1, 3.4.2, 4.1.1, 4.1.2, 4.1.3, 4.1.5, 4.1.6
4.1.7, 4.1.8, 4.1.9, 4.1.10, 4.1.13.

3.3.1.

the total periodic outlay is $L(i+\frac{1}{s_{\overline{n}|j}})$

$i=12\%,\quad n=10,\quad j=8\%.$

$$(X-12{,}000)\,s_{\overline{5}|8\%}\cdot(1+8\%)^5+(2X-12{,}000)\,s_{\overline{5}|8\%}=100{,}000 \Rightarrow X=13{,}454.36$$

3.3.2.

(a) $16{,}902.95 = L(10\%+\frac{1}{s_{\overline{10}|8\%}}) \Rightarrow L=100{,}000.$

(b) $K=L(i+\frac{1}{s_{\overline{n}|j}}) \Rightarrow L=\frac{K\,s_{\overline{n}|j}}{1+i\cdot s_{\overline{n}|j}}$

3.3.3.

$10{,}000=100\,s_{\overline{n}|0.09/12} \Rightarrow n=74.9,\quad 10{,}000=100\,\ddot{s}_{\overline{74}|0.0075}+X \Rightarrow X=81.67$

x is the payment in last year

Total paid is $10{,}000(\frac{0.15}{12})\times 75+\underbrace{100\times 74+81.67}=16{,}856.67$

3.3.4

(i) $\frac{250{,}000}{a_{\overline{10}|0.12}}=31{,}035.91$

(ii) $250{,}000(0.10+\frac{1}{s_{\overline{30}|j}}) = 31{,}035.91 \Rightarrow j=0.21322.$

3.4.1

$$PV=10{,}000(v^{12}+v^{24}+\cdots+v^{96})+800(v+v^2\cdots+v^{12})+700(v^{13}+\cdots+v^{24})+600(v^{25}+\cdots+v^{36})$$
$$+\cdots 400(v^{85}+\cdots+v)^{96}.\quad i=1\%.\quad v=\frac{1}{1+i}.$$

3.4.2

Makeham's formula $A=K+\frac{i}{j}(L-K)$

(a) $K=1000(v_{3\%}^{4}+v_{3\%}^{8}+\cdots v_{3\%}^{60})=6615.21.$

$PV=K+\frac{i}{j}(L-K)=6615.21+\frac{4\%}{3\%}(15{,}000-6615.21)=17.795.$

(b) $K=1000(v^4+2v^8+\cdots+5v^{20})=9832.49 \Rightarrow PV=16.723.$

(c) $K=1000(5v^4+\cdots+v^{20})=11{,}504.22 \Rightarrow PV=16{,}165$

4.1.1.

$P = C \times \frac{1}{(1+j)^n} + Fr\left(\frac{1}{1+j} + \frac{1}{(1+j)^2} + \cdots + \frac{1}{(1+j)^n}\right) = C v_j^n + Fr\, a_{\overline{n}|j}$.

$C = F$, $\underline{P = F v_j^n + Fr\, a_{\overline{n}|j}}$, $v_j^n = 1 - j\, a_{\overline{n}|j}$, $P = C + (Fr - Cj)\, a_{\overline{n}|j}$.

$C = F$ in this case. $P = F + F(r-j)\, a_{\overline{n}|j}$. j: 6-month yield rate. r: coupon rate

(a), (b), (c), (d) have same F, $j_{(a)} < j_{(b)}$, $\Rightarrow a_{\overline{n}|j(a)} > a_{\overline{n}|j(b)}$. $\Rightarrow (r_{(a)} - j_{(a)})\, a_{\overline{n}|j(a)} < (r_{(b)} - j_{(b)})\, a_{\overline{n}|j(b)}$

then $P_{(a)} < P_{(b)}$. Similarly, $P_{(c)} < P_{(d)}$.

4.1.2.

$P = 115.84$, $F = 100$, $r = \frac{7\%}{2}$, $j = \frac{6\%}{2}$, $n = 24$.

$P = C v_j^n + Fr\, a_{\overline{n}|j} = C v_{3\%}^{24} + 100 \times 3.5\%\, a_{\overline{24}|3\%} \Rightarrow C = 115$.

4.1.3.

$5083.49 = 10{,}000\, v_j^{20} \Rightarrow j = 0.0344$.

$X = 10{,}000\, v_j^{20} + 10{,}000 \times \frac{10\%}{2}\, a_{\overline{20}|j} = 12{,}229$.

4.1.5.

coupon payment: $1000 \times \frac{8\%}{2} = 40$.

accumulated value of reinvested coupon $40\, s_{\overline{20}|6\%/2} = 1074.81$.

After 10 yr, Don receives another redemption 1000, total is 2074.81.

then $900(1+j)^{20} = 2074.81 \Rightarrow j = 0.0426$, annual yield rate is 0.085.

4.1.6.

$P = 800$, $F = 1000$, $r = \frac{10\%}{4}$, $n = 25 \times 4 = 100$, $P = F v_j^{100} + Fr\, a_{\overline{100}|j} \Rightarrow 800 = 1000\, v_j^{100} + 25\, a_{\overline{100}|j} \Rightarrow j = 0.0316$

$4 \times 0.0316 = 0.1264$

4.1.7.

$F = 1000$, $r = \frac{10\%}{2} = 5\%$, $n = 10 \times 2 = 20$. yield rate $j = \frac{8\%}{2} = 4\%$.

$P = 1000\, v_{0.04}^{20} + 50\, a_{\overline{20}|0.04} = 1135.90$.

Loan repayment $1135.90(1+7\%)^{10} = 2234.49$.

reinvested coupons $50\, s_{\overline{20}|6\%/2} = 1343.52$. plus redemption $1343.52 + 1000$ minus 2234.49

$= 109.43$

as net gain.

4.1.8.

$r = \frac{4.78\%}{2}$, $j = \frac{4.855\%}{2}$, $F = 100$, $n = 5\times2 = 10$.

$P = 100 v_j^{10} + 100 r a_{\overline{10}|j} = 99.539$.

4.1.9.

(a) $j = \frac{3.69\%}{2}$, $Fr = \frac{4.25}{2} = 2.125$.

$100 v_j^{5.5\times2} + 2.125 a_{\overline{11}|j} = 102.76$.

(b) if $j = \frac{3.685\%}{2}$, $P = 102.79$, if $j = \frac{3.695\%}{2}$, $P = 102.74$.

(c) if $P = 102.765$, $j = 3.69\%$, if $P = 102.755$, $j = 3.692\%$.

4.1.10.

$P = F + F(r-j) a_{\overline{n}|j}$

$95.59 = 100 + 100(0.02375 - j) a_{\overline{20}|j}$,

$108.82 = 100 + 100(0.03125 - j) a_{\overline{20}|j}$

thus, $j = 0.02625$, $j^{(2)} = 0.0525$.

4.1.13.

By $P = F + F(r-j) a_{\overline{n}|j}$.

$79.30 = 100 + 100(7\%/2 - j) a_{\overline{2n}|j}$, ... ①

$93.10 = 100 + 100(9\%/2 - j) a_{\overline{2n}|j}$ $\Rightarrow j = 0.05$

plug back in ①, then $n = 24$. or 12 years.